AN INVESTIGATION OF THE PROPERTIES OF THE ENVELOPE ENERGY OF AN AMPLITUDE MODULATED NOISE CARRIER

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Abstract: The probability distributions of the amplitude of the envelope energy spectrum of both Gaussian noise and amplitude modulated Gaussian noise are derived. From these probability distributions the probability of detection of the modulation is calculated by comparing the distribution at the modulating frequency to the distribution at an adjacent frequency. Confidence percentages as a function of modulation factor are also calculated and compared with experimental results.

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CHAPTER I

INTRODUCTION

In recent years, the study of the characteristics of complex noise signals has become increasingly more important in the fields of communications and acoustics. Several mathematical surveys of this subject have been written by Rice [1]*, Davenport and Root [2], and others. Although these papers cover many of the properties of noise, none is involved with amplitude modulation of Gaussian noise which is an important consideration in the area of signal detection.

The surveys mentioned above contain the basic material needed for this study. Rice's definitions of narrow band Gaussian noise and the envelope of narrow-band Gaussian noise are used throughout this thesis. Of special interest is the treatment by both Rice and Davenport and Root of the output of square law detectors. If the input to such a device is narrow-band Gaussian noise, then the output is proportional to the square of the envelope. The authors' treatment [1,2] includes deriving the expression for the spectral density output of the detector for a Gaussian noise input.

In this study, the frequency spectrum distribution of the output of a square-law detector will be calculated. This distribution can then be related to the power spectral density

* Numbers in brackets are references in the Bibliography.

that has been derived in the literature. Then the probability density of the frequency spectrum will be derived for an input of amplitude modulated Gaussian noise to the square law detector. For this case, a spike will appear at the modulating frequency of the spectrum and the amplitude distribution of this spike will not be the same as the distribution of the spectrum at adjacent frequencies. The amplitude distribution of the spike will be derived separately and will be of primary importance in finding the probability of detection of the modulation.

The frequency distribution, once derived, will be modified to a special case so that the theoretical results can be compared to the results found by experiment. The modification is a function of the spectrum analyzer used for the experiment. For the experiments done in this study, a Quan Tech Wave Analyzer Model 304M will be used. The theoretical results will be modified so that they can be compared with the output of this particular analyzer.

From the final frequency distribution, it will be possible to establish a threshold level at the modulating frequency such that the probability of the output being above that threshold for unmodulated Gaussian noise is a set value. Using this threshold, the probability of the output being above that threshold for modulated noise can be found and plotted as a function of the percentage of modulation. It is these plots for different threshold levels that will be the main result of this research.

CHAPTER II

SPECTRAL DENSITY FOR A SQUARE LAW DETECTOR IN RESPONSE TO NARROW-BAND GAUSSIAN NOISE

Several articles have been written by various people, including Rice and Davenport and Root, in which an expression has been derived for the power spectrum of the square of the envelope of Gaussian noise. This is done by finding the power spectrum of the output of a system shown in Figure 2.1, consisting of a square law device followed by a low pass filter. If the input is narrow-band Gaussian noise, n(t), then according to Davenport and Root, n(t) can be expressed as:

$$n(t) = E(t) \cos [\omega_c t + \emptyset(t)] \quad , \qquad (1)$$

where E(t) is the envelope of the Gaussian noise, $f_c = \omega_c/2\pi$ is the center frequency of the input spectral density and $\emptyset(t)$ is the input phase. The output of the square law device would be:

$$S(t) = \frac{a E^2(t)}{2} + \frac{a E^2(t)}{2} \cos [2\omega_c t + 2\emptyset(t)] \qquad (2)$$

If the bandwidth of the noise is narrow compared with the center frequency, the spectral density of the two terms in Equation 2 will not overlap. Then if S(t) is passed through an ideal low pass filter, the output of that filter, z(t), will be:

$$z(t) = \frac{a E^2(t)}{2} \qquad (3)$$

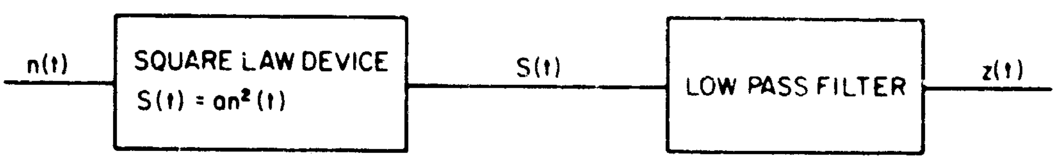

Figure 2.1 Square Law Detector

This can be normalized by making a = 2 so that the output of the filter will be equal to the square of the envelope.

Assume the spectral density of the narrow band Gaussian noise to be as shown in Figure 2.2. Then Davenport and Root [2] have found the spectral density of the output of the square law detector. This is given by the following:

$$S_z = 4a^2N_o^2W^2\,\delta(f) + 4a^2N_o^2W(1 - |f|/W) \qquad (4)$$

where N_o is the height of the spectral density of the input, W is the bandwidth of the input, and $\delta(f)$ is the Dirac delta function. S_z is shown in Figure 2.3 as a function of frequency.

It will be convenient in this study to further normalize the amplitude of E(t) so that the height of the continuous part of S_z at f = 0 is one. This will not change the results and will help simplify the equations derived. The constant factor would drop out when comparing the amplitude of adjacent frequencies of the frequency spectrum. To normalize, let

$$4a^2N_o^2W = 1 \qquad (5)$$

With this normalizing factor, Equation 4 can be rewritten as the following:

$$S_{zn} = W\,\delta(f) + (1 - |f|/W) \qquad (6)$$

When the input noise is modulated by a sine wave, the expression for the modulated noise would be of the form

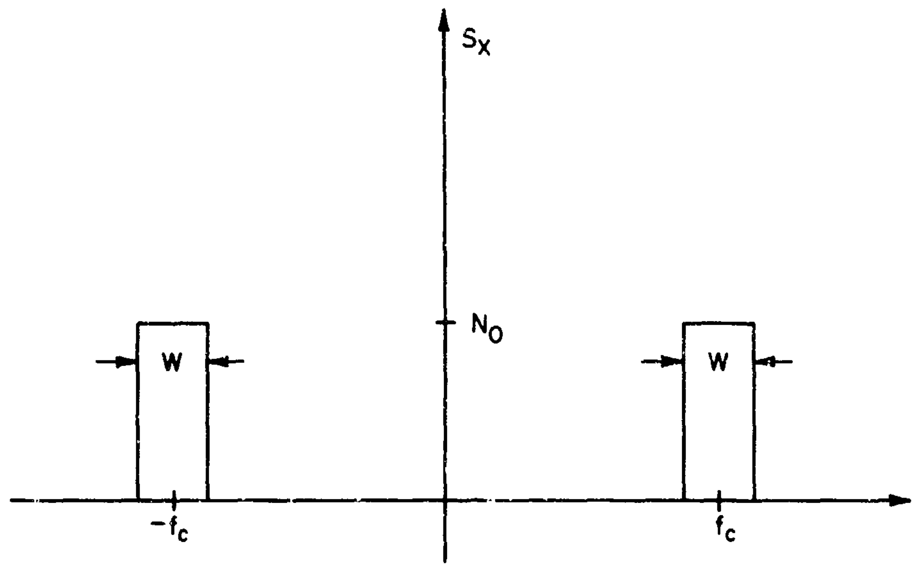

Figure 2.2 Spectral Density of Gaussian Noise

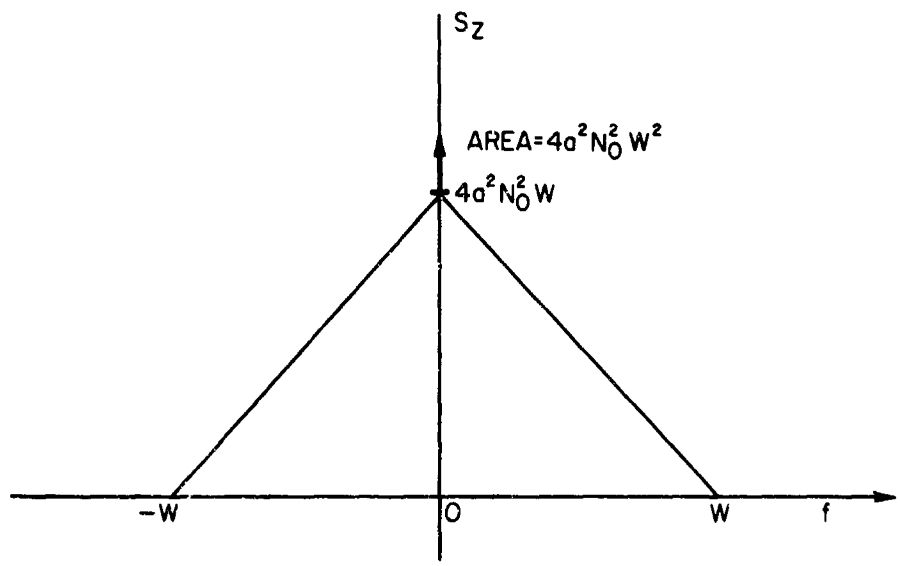

Figure 2.3 Spectral Density for a Square Law Detector Given a Narrow-Band Gaussian Noise Input

$(1 + m \cos \omega_m t)\, n(t)$, where m is the modulation factor and ω_m is the modulation frequency. It is then correct to say that the envelope of the noise is modulated in the same way. Therefore, the square of the envelope of the modulated noise is $(1 + m \cos \omega_m t)^2 E^2(t)$. The type of modulated noise that is of primary interest in this study is that of wide band modulated noise that is then passed through a narrow-band filter. The modulation produces sidebands in the power spectrum which will have impulses at plus and minus the modulating frequency. The problem is then to find the probability of the amplitude of this impulse being above a set threshold as a function of modulation factor.

CHAPTER III

THEORETICAL DISTRIBUTIONS

3.1 Unmodulated Gaussian Noise

If the Gaussian noise under consideration is of finite duration of time, then the amplitude of the energy spectrum of the envelope squared will be a random value with a certain distribution. If this amplitude distribution is found, then its variance can be related to the average power of the envelope squared as found in Chapter II. Then using a modified distribution which assumes amplitude modulated noise as an input, detection probabilities as a function of modulation factor can be calculated.

To begin, narrow-band Gaussian noise, $n(t)$, has been represented by Rice in the following way

$$n(t) = a(t) \cos \omega_c t - b(t) \sin \omega_c t \quad , \tag{7}$$

where $a(t)$ and $b(t)$ are both Gaussian functions of time and $f_c = \omega_c/2\pi$ is the midband frequency of the noise. The square of the envelope is then defined as the following:

$$E^2(t) = a^2(t) + b^2(t) \tag{8}$$

Noise of bandwidth W can be specified by taking samples at every $\frac{1}{2W}$ seconds. If a record of noise T seconds in duration is considered, then 2WT degrees of freedom are obtained by taking

the samples at intervals of $\frac{1}{2W}$ seconds. In representing the noise as in Equation 7, WT samples of a(t) and WT samples of b(t) will also specify the noise and give 2WT degrees of freedom. The amplitude of the samples of a(t) and b(t) will have a Gaussian distribution with zero mean and with variance σ_t^2. If the power of the noise in its passband is N_c, then $\sigma_t^2 = 2N_cW$.

The next step is to convert the samples of a(t) into the frequency domain. Let a(t) be represented by its samples as in Equation 9,

$$a(t) = a_1 \frac{\sin \pi Wt}{\pi Wt} + a_2 \frac{\sin \pi(Wt-1)}{\pi(Wt-1)} + \ldots\ldots + a_{WT} \frac{\sin \pi(Wt-WT+1)}{\pi(Wt-WT+1)} \tag{9}$$

where $a_1, a_2, \ldots\ldots, a_{WT}$ are the magnitudes of the WT samples of a(t). If A(f) represents the Fourier transform of a(t), then the Fourier transform of both sides of Equation 9 can be taken to obtain the following:

$$A(f) = \frac{1}{W} \left(a_1 + a_2 \cos 2\pi f/W - ja_2 \sin 2\pi f/W \right.$$
$$\left. + a_3 \cos 4\pi f/W - ja_3 \sin 4\pi f/W + \ldots \right) \tag{10}$$

The samples of A(f) are located at intervals 1/T over the band $-W/2 < f < W/2$. There are WT samples so that they located at $f = n/T$ where $n = -\frac{WT}{2}, -\frac{WT}{2}+1, \ldots, \frac{WT}{2}$. The value of $A(\frac{n}{T})$ from Equation 10 is:

$$A(\tfrac{n}{T}) = \frac{1}{W}\sum_{k=1}^{WT} a_k \cos\frac{(k-1)2\pi n}{WT} - \frac{j}{W}\sum_{k=1}^{WT} a_k \sin\frac{(k-1)2\pi n}{WT} \qquad (1)$$

The distribution of a sum of n Gaussian variables with zero mean and variance σ_t^2 is itself Gaussian with zero mean and variance equal to $n\sigma_t^2$. Therefore, the samples of $A(\frac{n}{T})$ have an amplitude distribution which is Gaussian. If σ_{fr}^2 represents the variance of the real part of $A(\frac{n}{T})$ and σ_{fi}^2 represents the variance of the imaginary part of $A(\frac{n}{T})$, then

$$\sigma_{fr}^2 = \frac{\sigma_t^2}{W^2}\sum_{k=1}^{WT} \cos^2\frac{(k-1)2\pi n}{WT}$$

$$= \frac{\sigma_t^2}{W^2}\sum_{k=1}^{WT}\left[\frac{1}{2} + \frac{1}{2}\cos\frac{(k-1)4\pi n}{WT}\right] = \frac{\sigma_t^2 T}{2W} = N_o T \qquad (2)$$

and

$$\sigma_{fi}^2 = \frac{\sigma_t^2}{W^2}\sum_{k=1}^{WT} \sin^2 (k-1)\frac{2\pi n}{WT}$$

$$= \frac{\sigma_t^2}{W^2}\sum_{k=1}^{WT}\left[\frac{1}{2} - \frac{1}{2}\cos (k-1)\frac{4\pi n}{WT}\right] = \frac{\sigma_t^2 T}{2W} = N_o T \qquad (3)$$

Now let α_n represent the samples of $A(\frac{n}{T})$ and β_n represent the samples that would be obtained in converting b(t) into the frequency domain. Then

$$\alpha_n = A_n \cos \omega_n t + j A'_n \sin \omega_n t \quad , \qquad (14)$$

$$\beta_n = B_n \cos \omega_n t + j B'_n \sin \omega_n t \quad , \qquad (15)$$

where A_n, A'_n, B_n, B'_n are Gaussian variables with zero mean and variance N_oT as found in Equations 12 and 13. To find the distribution of the envelope squared in the frequency domain, the distribution of $A(f)$ convolved with itself plus $B(f)$ convolved with itself must be found. Let $F(f)$ be the result of the sum of the convolutions. Then at a particular frequency n/T,

$$F\left(\frac{n}{T}\right) = \sum_{m=-\frac{WT}{2}}^{WT/2} [\alpha_m \alpha^*_{m-n} + \beta_m \beta^*_{m-n}] \qquad n \neq 0 \ , \qquad (16)$$

where * signifies the complex conjugate of the variable.

Then using Equations 14 and 15:

$$F\left(\frac{n}{T}\right) = \sum_{m=-\frac{WT}{2}}^{WT/2} [(A_m \cos \omega_m t + j A'_m \sin \omega_m t)(A_{m-n} \cos \omega_{m-n} t - jA'_{m-n} \sin \omega_{m-n} t)$$

$$+ (B_m \cos \omega_m t + jB'_m \sin \omega_m t)(B_{m-n} \cos \omega_{m-n} t - jB'_{m-n} \sin \omega_{m-n} t)]$$

$$= \frac{1}{2} \sum_{m=-\frac{WT}{2}}^{WT/2} \Big[(A_m A_{m-n} + A'_m A'_{m-n} + B_m B_{m-n} + B'_m B'_{m-n}) \cos(\omega_m - \omega_{m-n})t$$

$$+ (A_m A_{m-n} - A'_m A'_{m-n} + B_m B_{m-n} - B'_m B'_{m-n}) \cos(\omega_m + \omega_{m-n})t$$

$$+ j \Big\{ (A_m A'_{m-n} + A'_m A_{m-n} + B_m B'_{m-n} + B'_m B_{m-n}) \sin(\omega_m - \omega_{m-n})t$$

$$- (A_m A'_{m-n} - A'_m A_{m-n} + B_m B'_{m-n} - B'_m B_{m-n}) \sin(\omega_m + \omega_{m-n})t \Big\} \Big]$$

$$n \neq 0 \qquad (17)$$

Each product of variables has a distribution of the product of two Gaussians with zero mean and a variance

$$\sigma_m^2 = \frac{\sigma_t^4 T^2}{4W^2} + N_o^2 T^2 \quad . \qquad (18)$$

This is derived from the general theorem that states, if two independent variables have means, μ_1 and μ_2, and variance, σ_1^2 and σ_2^2, respectively, then the distribution of the product of the two variables has mean $\mu_1\mu_2$ and variance $\sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2$. In this case $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = N_o T$.

Equation 17 shows that $F(\frac{n}{T})$ consists of a sum of $16T(W - |n/T|)$ samples all with zero mean and variance $\frac{N_o^2 T^2}{4}$. By assuring ourselves that $16T(W - |n/T|)$ is a large number, the central limit theorem can be used to approximate the distribution of $F(\frac{n}{T})$. The only restriction in making $16T(W - |n/T|)$ large enough is to make

sure that the frequency of interest is near zero as compared with W or $-W$. The distribution of $F(\frac{n}{T})$ can be approximated by a Gaussian distribution with zero mean and a variance σ_n^2, where

$$\sigma_n^2 = 16T(W - |n/T|)\frac{N_o^2 T^2}{4} = 4 N_o^2 T^3 (W - \left|\frac{n}{T}\right|)$$

$$= 4 N_o^2 T^3 W (1 - |f|/W) \qquad (19)$$

Since a finite duration of time, T, is being dealt with, σ_n^2 is then equal to the average energy of the envelope squared in the bandwidth $1/T$. To normalize this variance, the energy must be related to the power spectrum shown in Figure 3. From Equation 6, the average power per cycle at a frequency f_x is $(1 - f_x/W)$. The power over $1/T$ cycles will then be $(1 - f_x/W)\frac{1}{T}$. Therefore, the energy in a bandwidth $1/T$ around the frequency f_x is $(1 - f_x/W)\frac{T}{T}$. Now Equation 19 can be normalized by setting σ_n^2 at f_x equal to $(1 - f_x/W)$.

$$4N_o^2 T^3 W (1 - f_x/W) = (1 - f_x/W)$$

$$N_o^2 = \frac{1}{4T^3W} \qquad (20)$$

Zero frequency is a special case and must be treated separately. Rewriting Equation 16 for $n = 0$ gives us:

$$F(0) = \sum_{m=-\frac{WT}{2}}^{WT/2} [\alpha_m \alpha_m^* + \beta_m \beta_m^*] \quad . \tag{21}$$

Again using Equations 14 and 15:

$$F(0) = \sum_{m=-\frac{WT}{2}}^{WT/2} [(A_m \cos \omega_m t + jA'_m \sin \omega_m t)(A_m \cos \omega_m t - jA'_m \sin \omega_m t)$$

$$+ (B_m \cos \omega_m t + jB'_m \sin \omega_m t)(B_m \cos \omega_m t - jB'_m \sin \omega_m t)]$$

$$= \frac{1}{2} \sum_{m=-\frac{WT}{2}}^{WT/2} [A_m^2 + A'^2_m + B_m^2 + B'^2_m + (A_m^2 - A'^2_m + B_m^2 - B'^2_m) \cos 2\omega_m t] \tag{22}$$

Since each variable is Gaussian, the distribution of the square of the variable is a chi-squared distribution with mean equal to $N_o T$ and variance $2N_o^2 T^2$. Again summing over these variables and applying the central limit theorem, we can approximate the distribution of $F(0)$ as a Gaussian distribution with a mean

$$\mu_o = 4WT \frac{\sigma_{fr}^2}{2} = 2 N_o T^2 W \tag{23}$$

and variance

$$\sigma_o^2 = 8WT \left(2 \frac{\sigma_{fr}^4}{4} \right) = 4T^3 N_o^2 W \quad . \tag{24}$$

This distribution can now be normalized with the help of Equation 20, to obtain

$$\mu_{on} = \frac{2T^2W}{\sqrt{4T^3W}} = \sqrt{WT} \tag{25}$$

and

$$\sigma_{on}^{\;2} = \frac{4T^3W}{4T^3W} = 1 \quad . \tag{26}$$

Now the amplitude distribution of the envelope squared in the frequency domain is specified at any frequency. The distribution is Gaussian at all frequencies with a mean of zero and variance of $(1 - |f|/W)$ at any non-zero frequency. At zero, the mean is $\sqrt{WT}$ and the variance is 1.

3.2 Amplitude Modulated Gaussian Noise

Now the distribution of the square of the envelope must be found given that the input noise is amplitude modulated by a sine wave of frequency f_m. The noise $n'(t)$ can be represented in the following way,

$$n'(t) = n(t) \quad (1 + m \cos \omega_m t) \quad . \tag{27}$$

Substituting for $n(t)$ from Equation 7,

$$n'(t) = a(t)\ (1 + m \cos \omega_m t) \cos \omega_1 t - b(t)\ (1 + m \cos \omega_m t) \sin \omega_1 t . \tag{28}$$

Then from Equation 8, if $E'(t)$ is the envelope of $n'(t)$,

$$[E'(t)]^2 = [a(t)\ (1 + m \cos \omega_m t)]^2 + [b(t)\ (1 + m \cos \omega_m t)]^2 \ . \tag{29}$$

Converting $a(t)\ (1 + m \cos \omega_m t)$ into the frequency domain will give $A(f)$ as in Equation 10 plus two sidebands of magnitude $A(f)\ m/2$. That is, if the samples of $A(f)$ have zero mean and variance $N_o T$ as in Equations 12 and 13, the samples of the sidebands will have zero mean and variance $N_o T\ m^2/4$.

Now let the samples of $A(f)$ be denoted by α_n and the samples of the upper sideband by α_{n1} and the samples of the lower sideband by α_{n2}. To find $[a(t)\ (1 + m \cos \omega_m t)]^2$ in the frequency domain, we must convolve the frequency spectrum of $a(t)\ (1 + m \cos \omega_m t)$ with itself. If $G(\frac{n}{T})$ is the result of this convolution, we will have:

$$G(\tfrac{n}{T}) = \sum_{m=-\frac{WT}{2}}^{WT/2} [\alpha_m \alpha^*_{m-n} + \alpha_{m1}\alpha^*_{m1-n} + \alpha_{m2}\alpha^*_{m2-n}$$

$$+ \alpha_{m1}\alpha^*_{m-n} + \alpha_m \alpha^*_{m1-n} + \alpha_{m2}\alpha^*_{m-n} + \alpha_m \alpha^*_{m2-n} + \alpha_{m1}\alpha^*_{m2-n} + \alpha_{m2}\alpha^*_{m1-n}] \ . \tag{30}$$

Proceeding similarly to the case with zero modulation, $G(\frac{n}{T})$ consists of $72T(W - |n/T|)$ samples with zero mean. Applying the central limit theorem to the sum, $G(\frac{n}{T})$ will be Gaussian with zero mean and variance

$$\sigma_g^2 = 8T(W - |n/T|)\left[\frac{N_o^2T^2}{4} + \frac{2N_o^2T^2m^4}{64} + \frac{4N_o^2T^2m^2}{16} + \frac{2N_o^2T^2m^4}{64}\right]$$

$$= 2N_o^2T^3W(1 - |f|/W)(1 + \frac{m^2}{2})^2 \quad . \tag{31}$$

Since the same procedure can be used for the samples of $b(t)$, the distribution of a sample of the square of the envelope in the frequency domain, $F'(f)$, will be Gaussian with zero mean and variance σ_m^2 equal to:

$$\sigma_m^2 = 4N_o^2T^3W(1 - |f|/W)(1 + \frac{m^2}{2})^2 \quad . \tag{32}$$

Now we must take into consideration that the convolution of the square of the envelope will give spikes at zero frequency, at the modulating frequency, and at twice the modulating frequency. The spike at zero arises when $A(f)$ is convolved with itself and when either sideband is convolved with itself. The spike at f_m arises when $A(f)$ is convolved with either sideband and the spike at $2f_m$ arises when one sideband is convolved with the other.

First, consider the spike at zero frequency,

$$G(0) = \sum_{m=-\frac{WT}{2}}^{WT/2} [\alpha_m\alpha_m^* + \alpha_{m1}\alpha_{m1}^* + \alpha_{m2}\alpha_{m2}^*$$

$$+ \alpha_{m1}\alpha_m^* + \alpha_m\alpha_{m1}^* + \alpha_{m2}\alpha_m^* + \alpha_m\alpha_{m2}^* + \alpha_{m1}\alpha_{m2}^* + \alpha_{m2}\alpha_{m1}^*] \quad . \tag{33}$$

The first three terms give distributions which are chi-squared, while the remaining terms have a distribution of the product of two

Gaussian variables. Summing over these variables and applying the central limit theorem, G(0) can be approximated by a Gaussian distribution with mean

$$\mu'_{om} = 2WT\left[\frac{N_oT}{2} + 2N_o\frac{Tm^2}{8}\right] = N_oT^2W\left(1 + \frac{m^2}{2}\right) \tag{34}$$

and variance

$$[\sigma'_{om}]^2 = 4WT\left[\frac{2N_o^2T^2}{4} + \frac{4N_o^2T^2m^4}{64}\right] + 8WT\left[\frac{4N_o^2T^2m^2}{16} + \frac{2N_o^2T^2m^4}{64}\right]$$

$$= 2N_o^2T^3W\left[1 + \frac{m^2}{2}\right]^2 \tag{35}$$

If we now take into account the terms for b(t), both the mean and variance of $F'(0)$ will be twice what was calculated in Equation 34 and 35. Therefore, the distribution of $F'(0)$ is Gaussian with mean

$$\mu_{om} = 2N_oT^2W\,(1 + m^2/2) \tag{36}$$

and variance

$$\sigma_{om}^2 = 4N_o^2T^3W\,(1 + m^2/2)^2 \quad . \tag{37}$$

For the spike at $f = f_m$, Equation 30 can be written as

$$G(f_m) = \sum_{m=-\frac{WT}{2}}^{WT/2} [\alpha_m\alpha^*_{m-f_mT} + \alpha_{m1}\alpha^*_{m1-f_mT} + \alpha_{m2}\alpha^*_{m2-f_mT}$$

$$+ \alpha_{m1}\alpha^*_{m-f_mT} + \alpha_m\alpha^*_{m1-f_mT} + \alpha_{m2}\alpha^*_{m-f_mT} + \alpha_m\alpha^*_{m2-f_mT} + \alpha_{m1}\alpha^*_{m2-f_mT} +$$

$$\alpha_{m2}\alpha^*_{m1-f_mT}] \quad . \tag{38}$$

The fifth and sixth terms will have a chi-squared distribution, while the other terms will have a distribution of the product of two Gaussian variables. Again applying the central limit theorem on the sum, $G(f_m)$ can be approximated as a Gaussian variable with mean

$$\mu'_{f_m} = \frac{2T}{2} \; (W-f_m) \left[\frac{2N_o Tm}{2}\right] = N_o W T^2 m \; (1 - f_m/W) \quad , \qquad (39)$$

and variance

$$\left[\sigma'_{f_m}\right]^2 = 4T(W-f_m)\left[\frac{4N_o^2 T^2 m^2}{16}\right] + 8T(W-f_m)\left[\frac{N_o^2 T^2}{4} + \frac{2N_o^2 T^2 m^4}{64} + \frac{2N_o^2 T^2 m^2}{16} + \frac{2N_o^2 T^2 m^4}{64}\right]$$

$$= 2N_o^2 T^3 W \; [1 + \frac{m^2}{2}]^2 \; [1 - f_m/W] \quad . \qquad (40)$$

If the terms for b(t) are taken into account, both the mean and variance of $F'(f_m)$ will be twice what was calculated to Equations 39 and 40. Therefore, the distribution of $F'(f_m)$ is Gaussian with mean

$$\mu_{f_m} = 2N_o W T^2 m \; [1 - f_m/W] \quad , \qquad (41)$$

and variance

$$\sigma^2_{f_m} = 4N_o^2 T^3 W \; [1 + \frac{m^2}{2}]^2 \; [1 - f_m/W] \quad . \qquad (42)$$

Since the variance in Equation 32 must now be normalized as in the case of modulation, a similar procedure is used to obtain

a value for N_o^2. Referring to Equation 20, the following equation is obtained,

$$4N_o^2T^3W\,(1 - f/W)\,(1 + m^2/2)^2 = (1 - f/W)$$

$$N_o^2 = \frac{1}{4T^3W(1 + m^2/2)^2} \quad . \tag{43}$$

Therefore, the distribution of a sample of a frequency f_m will be Gaussian with mean

$$\mu_s = \frac{m\sqrt{WT}\,(1 - f_m/W)}{(1 + m^2/2)} \quad , \tag{44}$$

and variance

$$\sigma_s^2 = (1 - f_m/W) \quad . \tag{45}$$

CHAPTER IV

EXPERIMENTAL ANALYSIS

The information given by Equations 44 and 45 is sufficient for finding the probability of detection of the modulation as a function of the modulation factor. However, to verify these equations, they must be modified so that the experimental results can be compared with the theoretical results. The modification of these equations will depend on what equipment is used to find the experimental values and this can be seen best by explaining the experimental setup. A block diagram of the system used is shown in Figure 4.1.

A modulating frequency of 50 Hz was used to modulate broadband noise. The modulated noise was then passed through a 2-4 kHz octave band filter and amplified. To obtain the square of the envelope, the modulated noise was rectified, squared, and passed through a low pass filter. The resulting signal was then converted into the frequency domain by use of a Quan-Tech Wave Analyzer. A one-cycle bandwidth filter in the analyzer was used in keeping with the equations derived in Chapter III. The analyzer was set on the frequency of interest, 50 Hz, and the amplitude of the output of the analyzer at that frequency was recorded on an X-Y plotter for 500 seconds. The amplitude distribution of the output of the analyzer was found by plotting a histogram of the amplitude at every half-second for the 500 seconds. The curve became

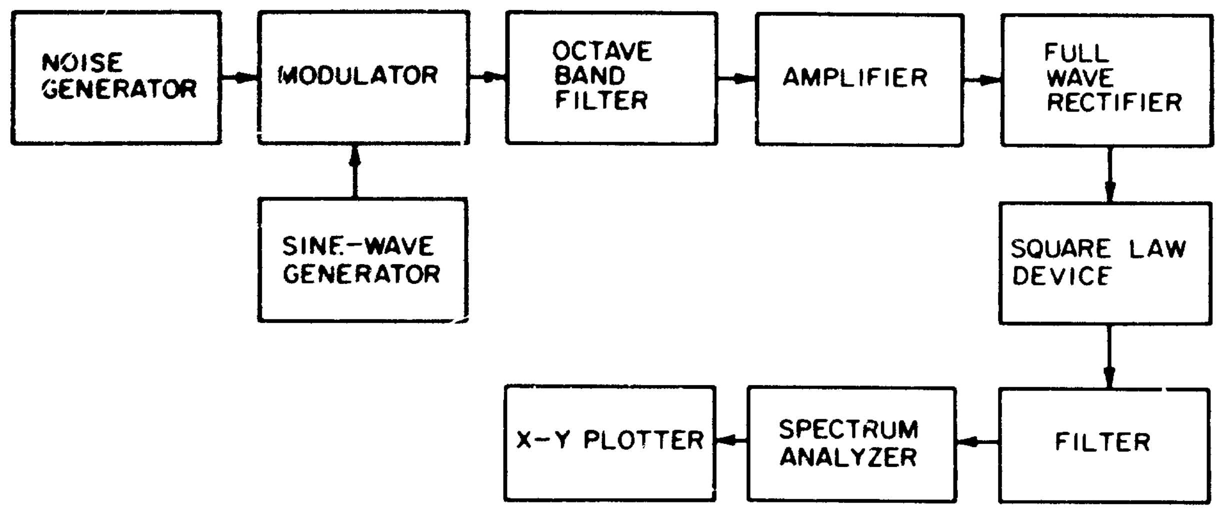

Figure 4.1 Block Diagram of Experimental Setup

sufficiently smooth after using 1000 points so that it could be compared with a theoretical curve.

A 50-second sample of an X-Y recording is shown in Figure 4.2. In this sample, no modulation was present on the input signal. Since the amplitude of the output voltage ranged from zero to 0.32 volts over the entire 500 seconds, amplitude increments of 0.02 volts were used to plot the histogram. For each modulation factor, ten samples of data similar to that shown in Figure 4.2 were used to find the amplitude distribution of the output of the analyzer

The theoretical distribution can be found, knowing how the spectrum analyzer treats the signal. In this case, the output of the one-cycle filter in the analyzer is passed through a voltage doubler and a low pass filter In effect, this gives the envelope of the signal coming out of the filter. In Chapter III, the output of the one-cycle filter was found to have a Gaussian distribution with mean μ_s given by Equation 44 and variance σ_s^2 given by Equation 45. If there is no modulation present, μ_s is zero and the distribution of the Gaussian noise will be a Rayleigh distribution. When modulation is introduced, the mean μ_s becomes non-zero and the distribution of the envelope will be a function of μ_s Rice has derived the distribution of the envelope of Gaussian noise plus a D.C. voltage. In this case, the D C voltage is the mean of the Gaussian distribution The distribution of the envelope of Gaussian noise with mean μ_s and variance σ_s^2, according to Rice is the following:

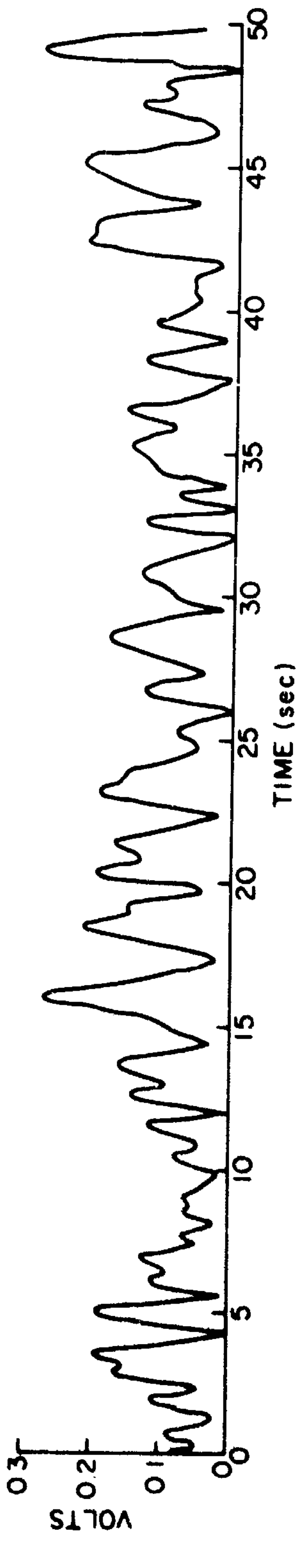

Figure 4.2 X-Y Recording of Analyzer Output

$$p\,(E) = \frac{E}{\sigma_s^2} \exp\left[-\frac{E^2 + \mu_s^2}{2\sigma_s^2}\right] I_o\left(\frac{E\mu_s}{\sigma_s^2}\right), \qquad (46)$$

where $I_o(x)$ is the modified Bessel function of zero order and can be conveniently found in the Chemical Rubber Company Tables [5].

To normalize the above equation with the distributions found experimentally, the mean of the above equation for $\mu_s = 0$ was set equal to the mean found in the experimental distribution for $m = 0$. The mean of Equation 46 for $\mu_s = 0$ is 1.25 σ_s so that the value of σ_s can be calculated. The experimental mean for the distribution was found to be 0.118v. so that $\sigma_s = 0.0944$ and $\sigma_s^2 = 0.0089$. Knowing this value and calculating μ_s from Equation 44 for a given value of m and normalizing μ_s, Equation 46 can be plotted and compared to the distributions found experimentally. Five different modulation factors of m = 0, 0.025, 0.05, 0.075, and 0.1 were used in the experiments and the plots of the distributions for each of these modulation factors is shown in Figures 4.3 - 4.7. The indicated modulation factors for the experimental distributions were found by using the mean of the distribution to determine what value of m would give that mean theoretically. These indicated modulation factors are well within the accuracy of the equipment used for the experiment.

With the information available from the preceding distributions, confidence percentages can be calculated for a given

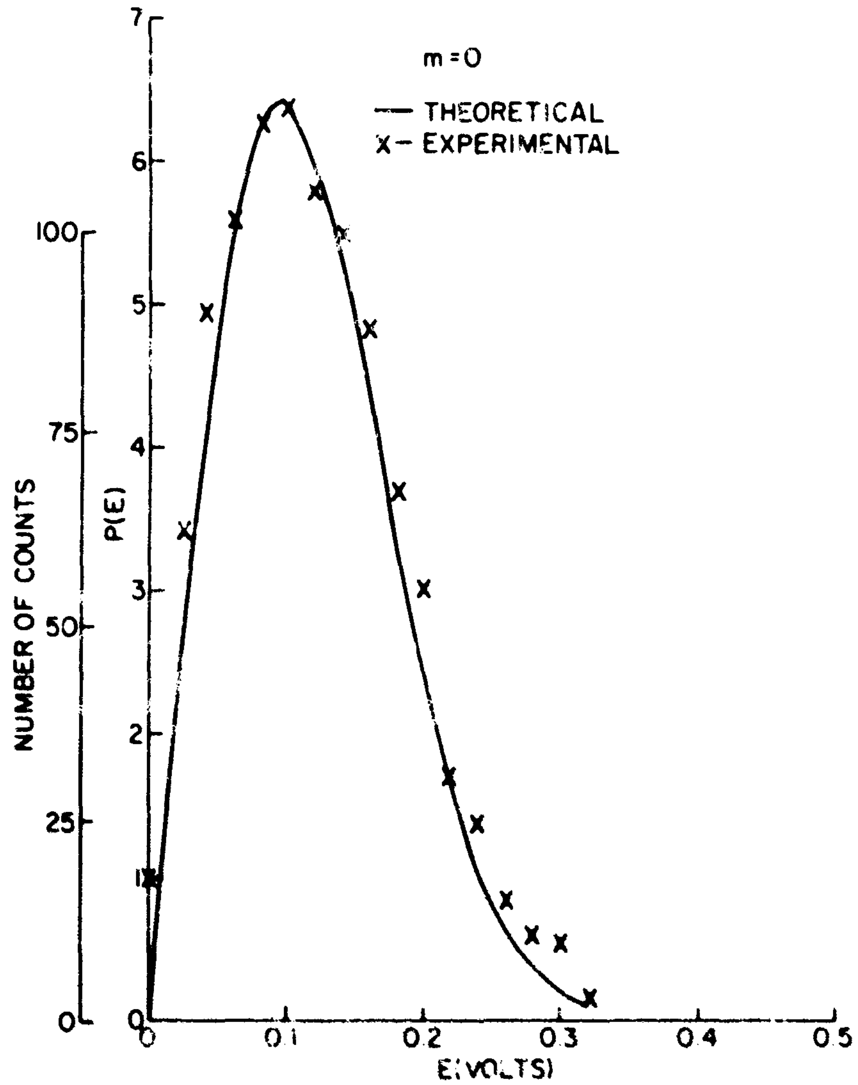

Figure 4.3 Analyzer Output Distribution for m = 0

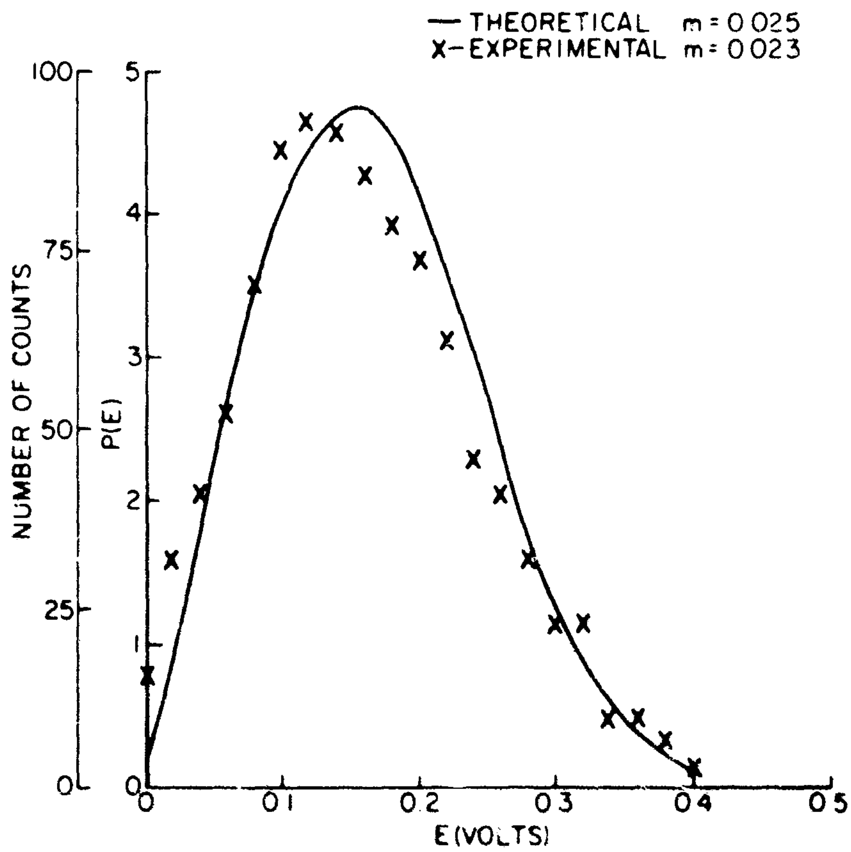

Figure 4.4 Analyzer Output Distribution for m = 0.025

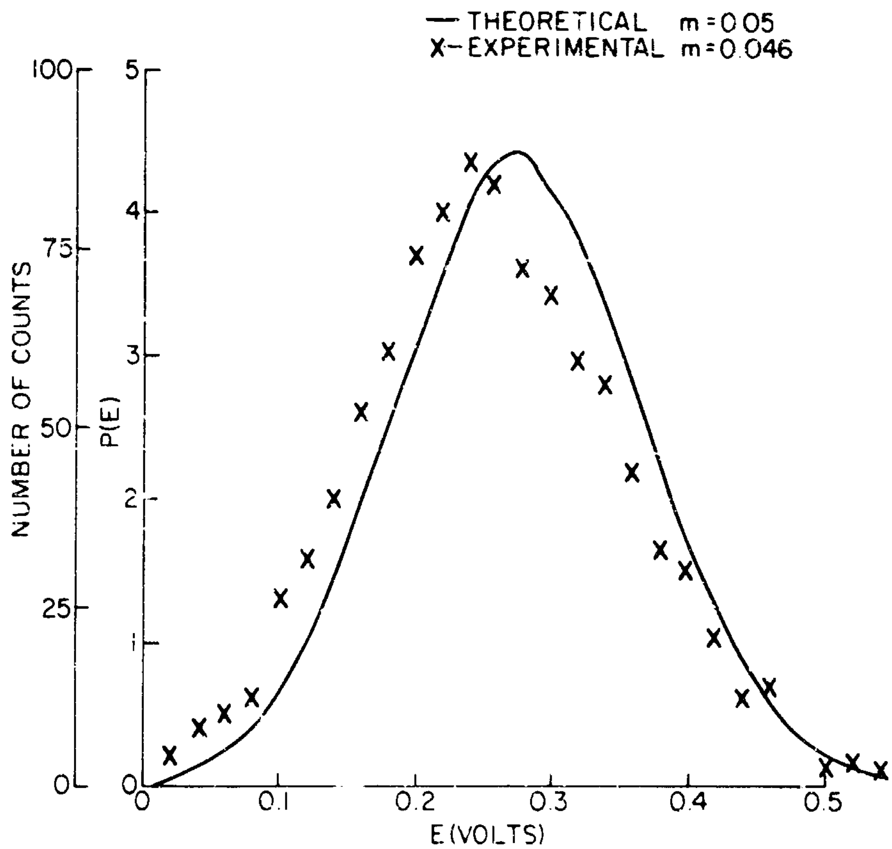

Figure 4.5 Analyzer Output Distribution for m = 0.05

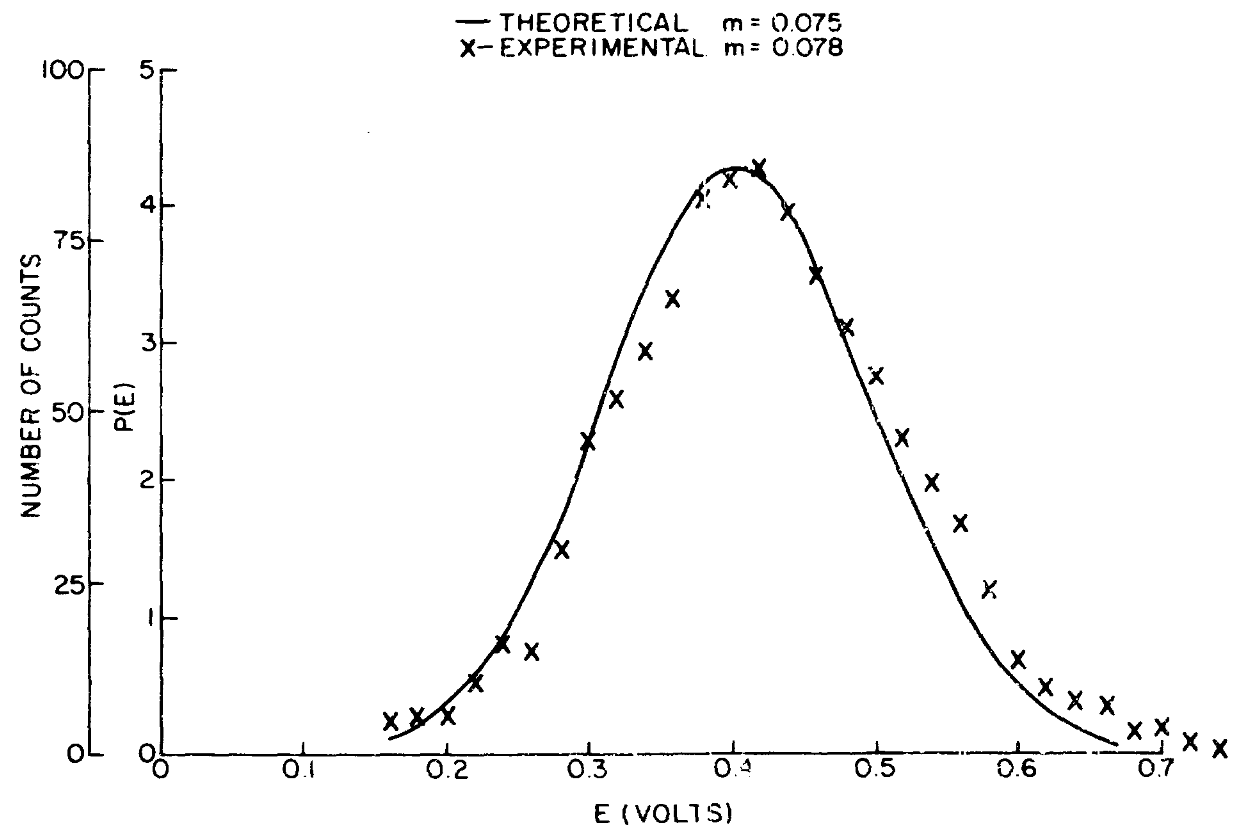

Figure 4.6 Analyzer Output Distribution for m = 0.075

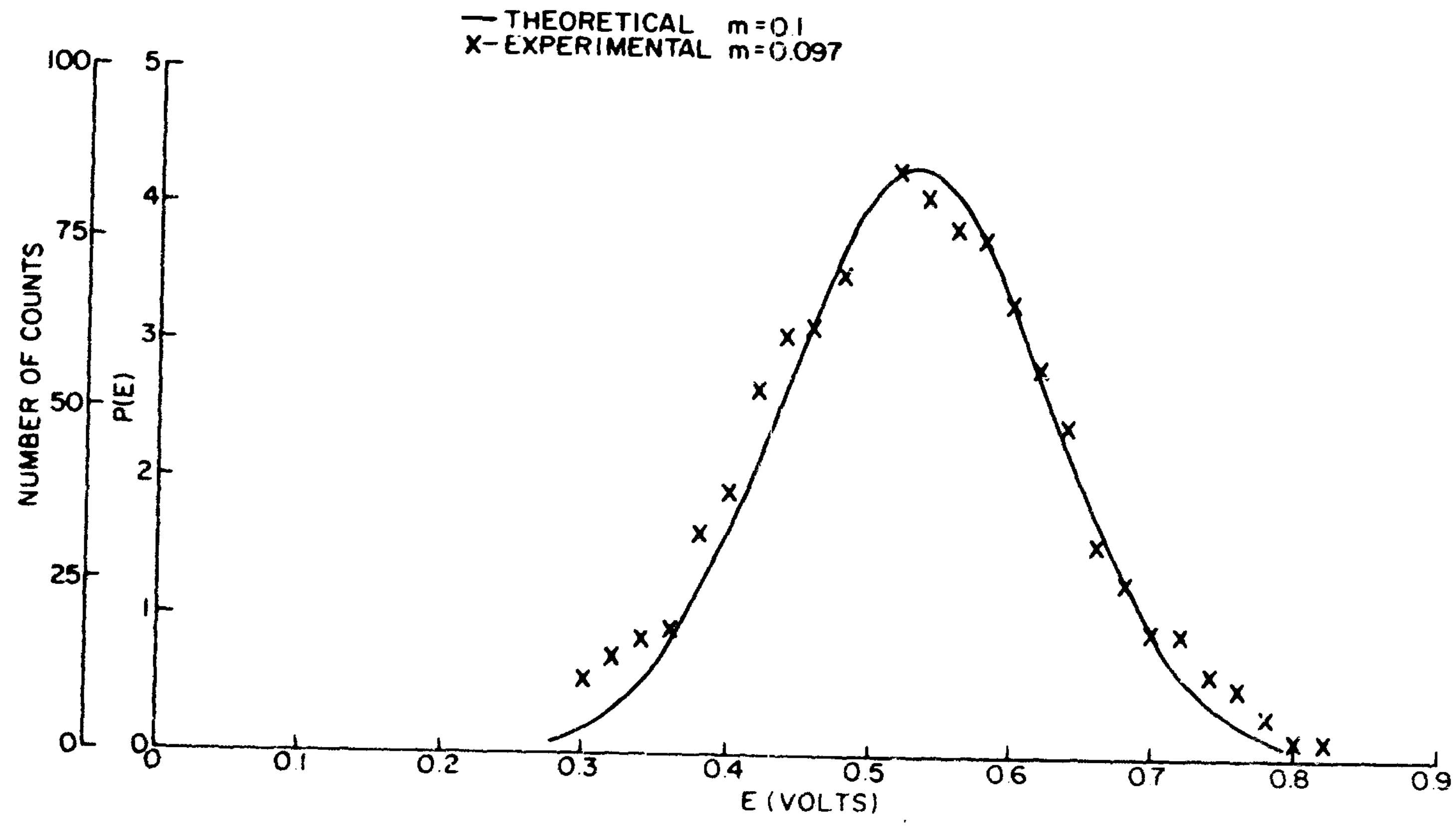

Figure 4.7 Analyzer Output Distribution for m = 0.1

modulation factor. The confidence percentage is defined as the probability of detecting the modulation for a given threshold level. The threshold level can be calculated for a given probability of allowing the signal to exceed the threshold for zero modulation. Since the amplitude distribution of the signal for zero modulation was found to be a Rayleigh distribution, the thresholds can be calculated from the following equation:

$$\int_0^t \frac{E}{\sigma_s^2} e^{-E^2/2\sigma_s^2} \, dE = 1 - p \quad , \qquad (47)$$

where t is the threshold and p is the probability of the signal being above t. The left-hand side of Equation 47 is integrable in closed form and becomes:

$$1 - e^{-t^2/2\sigma_s^2} = 1 - p$$

$$e^{-t^2/2\sigma_s^2} = p \qquad (48)$$

Since σ_s^2 is known to be 0.0089, values of t can be calculated for given values of p and these values are shown in Table 4.1.

For each of these thresholds, a confidence percentage can be calculated by integrating Equation 46 from t to ∞. This will give the probability that the modulated signal is above the threshold t. Since the expression for the distribution cannot be integrated readily, the integration was done numerically by the use of Simpson's rule. Figures 4.8 - 4.11 show the result of this integration as

TABLE 4.1

PROBABILITY OF THE SIGNAL BEING ABOVE THE THRESHOLD

Threshold t	Probability of Signal Being Above t
0.28	0.01
0.22	0.05
0.20	0.10
0.16	0.25

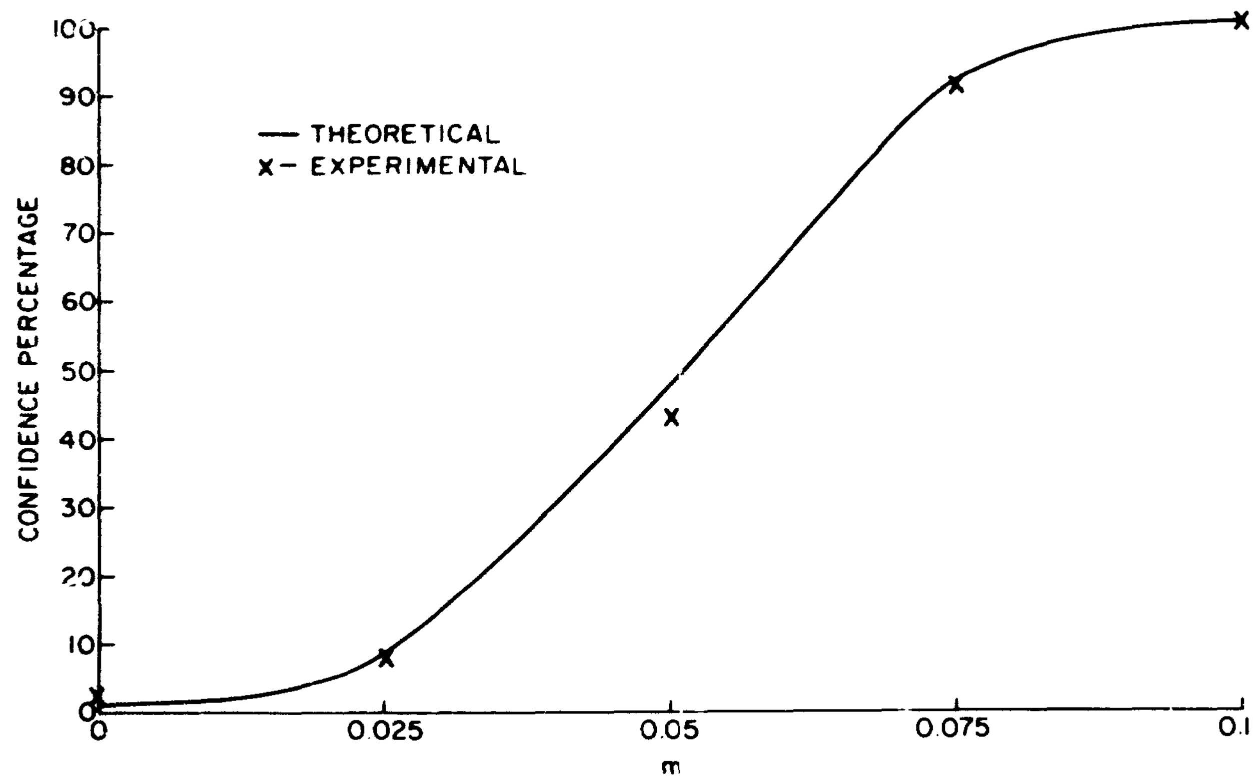

Figure 4.8 Confidence Percentages for t = 0.28 v.

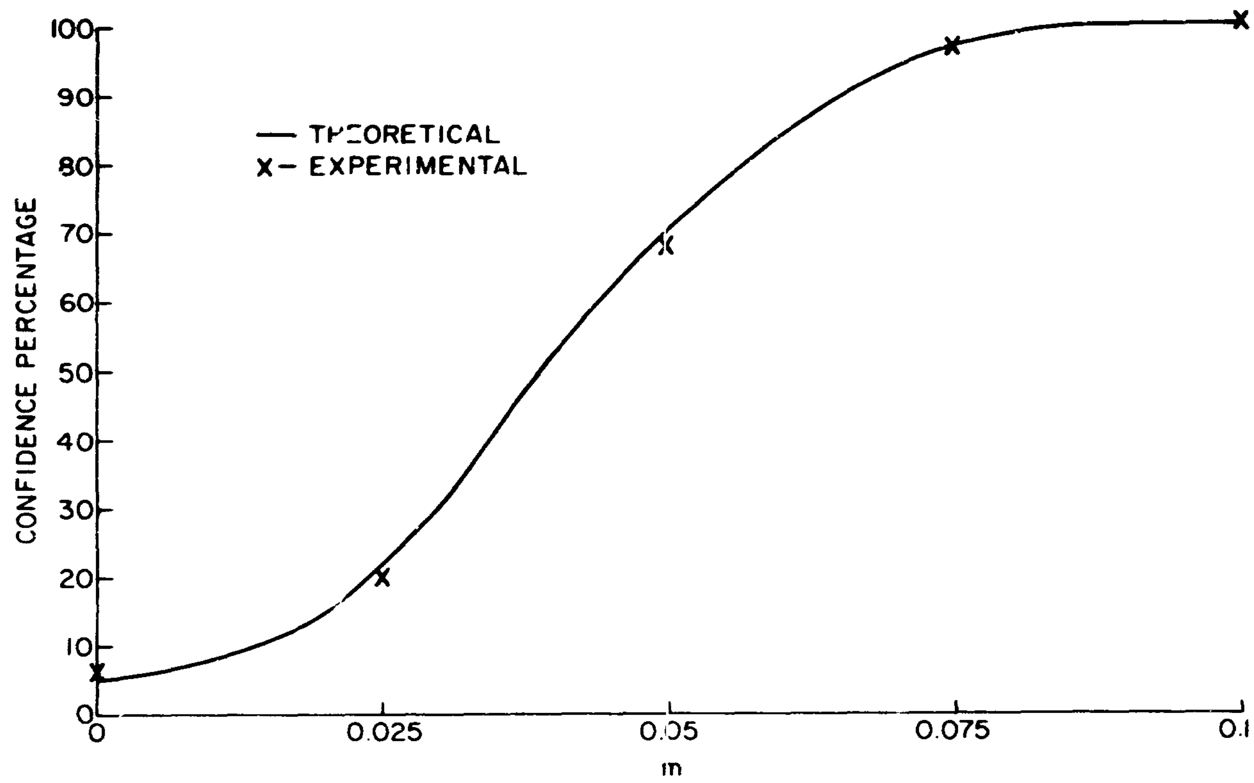

Figure 4.9 Confidence Percentages for t = 0.22 v.

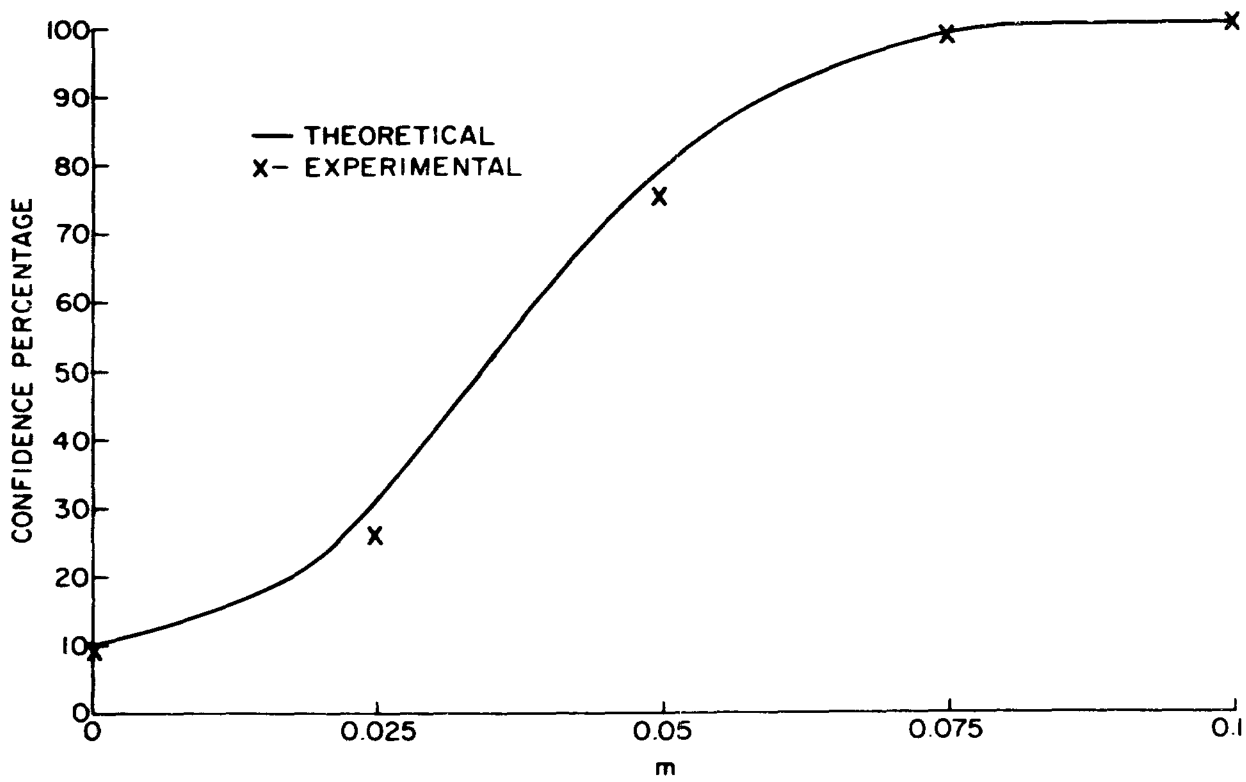

Figure 4.10 Confidence Percentages for t = 0.20 v.

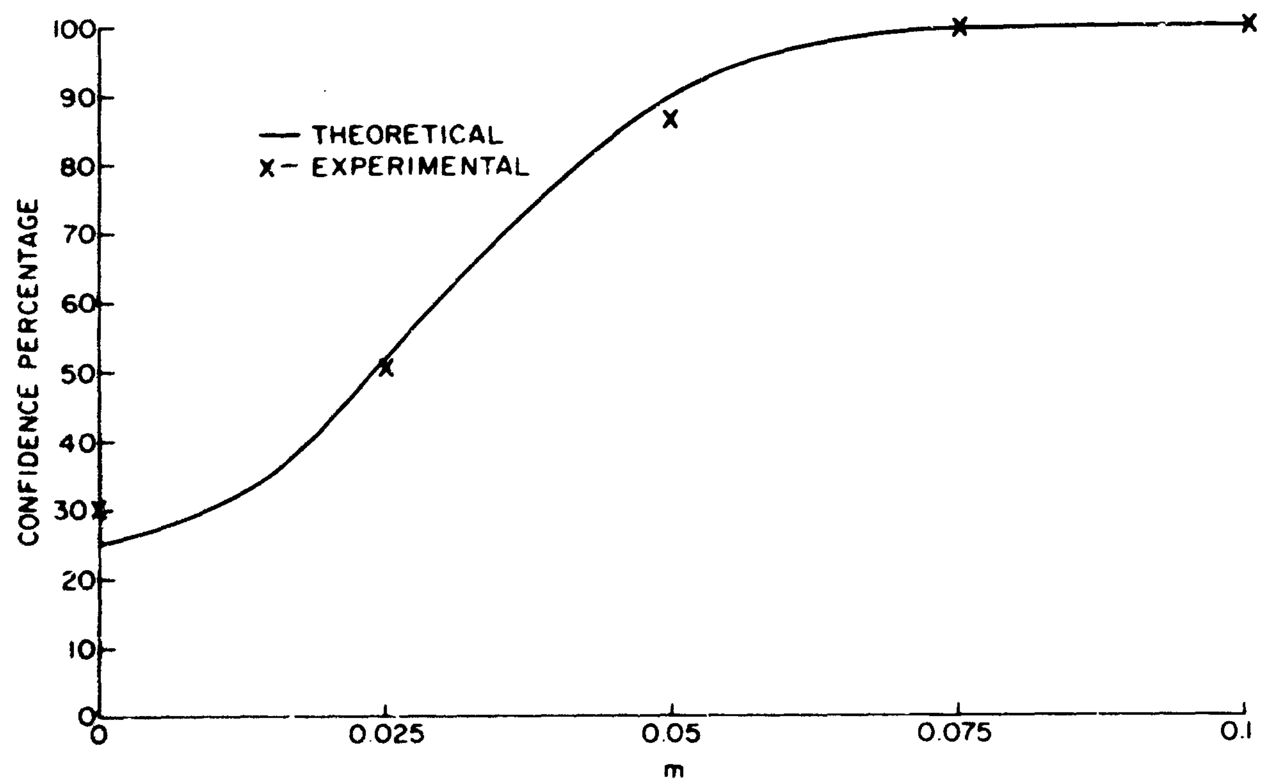

Figure 4.11 Confidence Percentages for t = 0.16 v.

a function of modulation factor. The experimental results were found from the histograms by counting the number of samples above the threshold and dividing by 1000. This gives an approximate value for the area of the curve above the threshold. From these plots, the percentage of the time the signal is above a given threshold can be found as a function of the modulation factor.

It must be remembered that the experimental results in this study are obtained for noise of a particular bandwidth of 2 kHz. It can be seen from Equation 44 that the mean value of the spike at the modulating frequency is proportional to the square root of the noise bandwidth. Suppose the noise was of bandwidth 20 kHz. This would increase the mean value of the spike by a factor of 3.16 and would make the confidence percentages even higher since the output of the analyzer would be above the threshold more of the time. Likewise, the confidence percentage for a given modulation factor would be less if the bandwidth of the noise is less than the 2 kHz used in the experiment. The confidence percentages for different bandwidths can be found by replotting Equation 46 with the new value for the mean and by finding the area under the curve above the threshold value.

CHAPTER V

SUMMARY

5.1 Results and Conclusions

An important result of this study is found in the curves of Chapter IV showing the percentage of time the frequency spectrum of the envelope of modulated Gaussian noise is above a certain threshold at the modulating frequency. It can be seen from these curves that 10 percent modulation will almost always be above a threshold for which a signal with no modulation would only be above one percent of the time. This means that the spike at the modulating frequency in the frequency spectrum will be easily detectable above the adjacent frequencies for any noise of bandwidth 2 kHz that is modulated at least 10 percent by a sine-wave. Of course, lowering the threshold will increase the chance of detecting the spike but that also will increase the false alarm rate. That is, there will be a greater chance of the signal being above the threshold when there is no modulation present.

A more general result that has been found is that the distribution of the frequency spectrum for the envelope of modulated Gaussian noise is a Gaussian distribution with mean μ_s given by Equation 44 and σ_s^2 given by Equation 45. In this study, the spectrum analyzer used gives the envelope of this frequency spectrum but this is not always the case. For example, several

spectrum analyzers use a square law detector at the output to give the energy spectrum. This would give an output entirely different from the one that was obtained in this study. In any case, knowledge of how the spectrum analyzer treats the signal is essential in determining the amplitude distribution of the output of the analyzer and this distribution must be found so that the theoretical and experimental results can be compared.

5.2 Areas for Further Study

One area for further study would be to find the probability of detection of modulation by using the amplitude distribution of the modulated Gaussian noise rather than the spectral distribution as used in this study. The results could then be compared with the probabilities of detection found in this study.

Another area for study would be to consider several other types of modulation beside sine wave modulation. Amplitude modulation of a noise carrier from a radiated noise source may not always be sine wave modulated. The modulation might be closer to square wave modulation or even triangular wave modulation. These types of modulations should be studied and compared with the results of this study which involves only sine wave modulation.

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